q-deformed Chern class, Chern-Simons and cocycle hierarchy

1995 J. Phys. A: Math. Gen. 28 543

(http://iopscience.iop.org/0305-4470/28/3/011)

J. Phys. A: Math. Gen. 28 (1995) 543–558. Printed in the UK

q-deformed Chern class, Chern–Simons and cocycle hierarchy

Bo-Yu Hou†, Bo-Yuan Hou‡ and Zhong-Qi Ma§

† Institute of Modern Physics, Northwest University, Xi'an 710069, People's Republic of China
‡ Graduate School, Chinese Academy of Sciences, PO Box 3908, Beijing 100039, People's Republic of China
§ CCAST (World Laboratory), PO Box 8730, Beijing 100081, People's Republic of China and Institute of High Energy Physics, PO Box 918(4), Beijing 100039, People's Republic of China

Received 28 March 1994, in final form 22 August 1994

Abstract. Based on the q-deformed BRST algebra of Bernard and Watamura, we study the q-deformed Killing form and the second q-deformed Chern class for the quantum group $SU_q(2)$ from the q-gauge covariant condition. We find that although the components of the identity and the adjoint representations are mixed in the covariant commutative relations of the q-deformed BRST algebra, the q-deformed Chern class can be defined uniquely up to a common factor such that it contains only adjoint components. We compute the q-deformed Chern–Simons by introducing a q-deformed homotopy operator, that is the quantum analogue of the homotopy operator presented by Chern and Zumino. Finally, we calculate the q-deformed cocycle hierarchy.

1. Introduction

Recently, quantum groups have attracted increasing attention. Manin [1] suggested a general construction for quantum groups as linear transformations on the quantum superplane. Following the general ideas in Connes [2] on the non-commutative geometry, Woronowicz [3] elaborated the framework of the non-commutative differential calculus. He introduced the bimodule over the quantum group and presented various theorems concerning the differential forms and exterior derivative. The differential calculus on the quantum hyperplane was developed by Wess and Zumino [4]. There have been many papers treating the differential calculus on quantum groups and the q-deformed gauge theories from various viewpoints [5–15].

Brzeziński and Majid [11] took some steps towards developing a gauge theory in which the quantum groups appear as the fibre of a quantum principal bundle and play the role of the structure group in the group of gauge transformations. Some physicists have tried to study the q-deformation of gauge theory from the interaction between matter fields and gauge fields, where the gauge fields are valued in the quantum groups but spacetime is an ordinary manifold. Some proposals [9] studied the covariance of the q-deformed Yang–Mills theory in the finite gauge transformation without considering the matter fields. These proposals were criticized [16] owing to the non-invariance of the so-called q-trace under repeated gauge transformations.

Since the quantum group is formulated in the language of the Hopf algebra, the gauge transformation will be represented in an abstract language and the term for the transformation parameter becomes obscure. Bernard [5] first raised the idea of q-deformed BRS symmetry

[16] as an alternative formulation of the gauge theory. Watamura [15] explicitly constructed a q-deformed BRST algebra, which is the algebra of the gauge fields, the ghost fields and appropriate matter fields on one spacetime point. Now, the gauge transformation of the theory is replaced by the BRST transformation which is represented by a nilpotent 'differential operator' δ, and the gauge parameter is replaced by the ghost fields and becomes an object of equal importance with the matter and gauge fields. Watamura [15] extracted appropriate properties from the non-deformed BRST formalism and imposed them as the condition which the new algebra should satisfy so that the q-deformed BRST formalism is the q-analogue of the non-deformed one. Watamura proved in detail that the definitions of the covariant commutation relations among the fields and their derivatives are consistent with the operation δ, the spacetime derivative d, as well as the *-operation, the antimultiplicative inner involution. Since there are two nilpotent operators δ and d, the double cohomology and cocycle hierarchy can be discussed in this formalism. The q-deformed BRST algebra of Watamura is a good starting point for developing the deformation of Chern classes and cocycle hierarchy.

The q-deformed Killing form is the key to constructing the q-deformed Chern class and the q-deformed cocycle hierarchy. The q-deformed Killing form and the second q-deformed Chern class, that contain both components of the identity and adjoint representations, for the quantum group $SU_q(2)$ can be defined in the q-deformed BRST formalism from the q-gauge covariant condition. Although the components of the two representations have to be mixed in the commutative relations of the q-deformed BRST algebra, we prove that it is possible to define the q-deformed Killing form and the second q-deformed Chern class uniquely up to a common factor such that they only contain the components of the adjoint representation. Then, generalizing Zumino's method [18, 19], we introduce a q-deformed homotopy operator to compute the q-deformed Chern–Simons, that also only contains adjoint components. Finally, by making use of the standard method for a non-deformed case, we calculate the q-deformed cocycle hierarchy. The formalism discussed in this article can be generalized to the quantum groups $SU_q(N)$.

Aschieri and Castellani [13] gave a pedagogical introduction to the differential calculus on quantum groups by stressing, at all stages, its connection with the classical case ($q \to 1$). In an article on deformed gauge theories, Castellani [14] tried to construct the q-deformed Lagrangian and the q-deformed Killing form. He found the q-deformed Killing form for $U_q(2)$ with a parameter, that may describe the mixture of the components of two representations. However, he did not find the general forms for the quantum groups $SU_q(N)$.

The plan of this article is as follows. In section 2 we sketch some formulae for a non-commutative differential calculus on $SU_q(2)$ [2–6, 8, 13] and the q-deformed BRST formalism [15] in order to explain our notation. The q-deformed Killing form g_{IJ} is defined in section 3 by requiring the invariance of the second q-deformed Chern class. g_{IJ} can be defined uniquely up to a common factor from the requirement that it only contains the adjoint components. In section 4 we introduce a q-deformed homotopy operator, that is the analogue of the homotopy operator presented by Chern and Zumino [18, 19], to compute the q-deformed Chern–Simons, from which we calculate the q-deformed cocycle hierarchy in section 5. We directly prove the cocycle hierarchy formulae by the recursive relations given in the appendix. Finally, some conclusions and discussions are given in section 6.

2. Non-commutative differential geometry and BRST formalism

A quantum group is introduced as the non-commutative Hopf algebra $\mathcal{A} = Fun_q(G)$ obtained by continuous deformations of the Hopf algebra of the function of a Lie group. The associated algebra $\mathcal{A}$ is freely generated by non-commutating matrix entries $T^a_{\ b}$ satisfying the relation

$$\left(\hat{R}_q\right)^{ab}_{\ \ ef} T^e_{\ c} T^f_{\ d} = T^a_{\ e} T^b_{\ f} \left(\hat{R}_q\right)^{ef}_{\ \ cd} \tag{2.1}$$

where $\hat{R}_q$ is the well known solution of the simple Yang–Baxter equation [20], related to the fundamental representation of $U_q SU(2)$ [21]:

$$\hat{R}_q = q\mathcal{P}_S - q^{-1}\mathcal{P}_A \qquad \hat{R}_q^{-1} = q^{-1}\mathcal{P}_S - q\mathcal{P}_A \tag{2.2}$$

where $\mathcal{P}$ denotes the projection operator. Hereafter, unless specified otherwise, summation of the repeated indices is understood.

The direct product representation of the fundamental representation and its conjugate

$$M^{a_1 b_2}_{\ \ a_2 b_1} = T^{a_1}_{\ b_1} \kappa\left(T^{b_2}_{\ a_2}\right) \tag{2.3}$$

contains both components of the identity and the adjoint representations, that can be separated by q-Pauli matrices [15]:

$$\begin{aligned}
&q^{-1}[2]^{1/2}\ (\sigma^0)_{ab} = q[2]^{1/2}\ (\sigma_0)^{ab} = -\epsilon^{ab} = \epsilon_{ab} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \\
&(\sigma^3)_{ab} = (\sigma_3)^{ab} = -[2]^{-1/2} \begin{pmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix} \\
&(\sigma^+)_{ab} = (\sigma_+)^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad (\sigma^-)_{ab} = (\sigma_-)^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&(\sigma^I)_a^{\ b} = (\sigma^I)_{ad}\epsilon^{bd} \qquad (\sigma_I)^a_{\ b} = (\sigma_I)^{ad}\epsilon_{db} \\
&(\sigma^I)_{ab}(\sigma_J)^{ab} = \delta^I_J \qquad (\sigma^I)_{ab}(\sigma_I)^{cd} = \delta^c_a\ \delta^d_b \\
&(\sigma^I)_a^{\ b}(\sigma_J)^a_{\ b} = \delta^I_J \qquad (\sigma^I)_a^{\ c}(\sigma_I)^b_{\ d} = \delta^b_a \delta^c_d \\
&(\mathcal{P}_A)^{ab}_{\ \ cd} = (\sigma_0)^{ab}(\sigma^0)_{cd} \qquad (\mathcal{P}_S)^{ab}_{\ \ cd} = (\sigma_i)^{ab}(\sigma^i)_{bd}
\end{aligned} \tag{2.5}$$

where the q-number is defined as usual:

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{2.6}$$

Throughout this paper, a capital italic letter, such as I, runs over 0, +, 3 and −, a small italic letter, such as i, runs over +, 3 and −, but the first few small italic letters, such as a, runs over 1 and 2. Now, two kinds of component can be separated in M:

$$\begin{aligned}
&M^I_{\ J} = (\sigma^I)_{a_1}^{\ a_2} M^{a_1 b_2}_{\ \ a_2 b_1} (\sigma_J)^{b_1}_{\ b_2} \\
&M^i_{\ 0} = M^0_{\ i} = 0 \qquad M^0_{\ 0} = \mathbf{1}.
\end{aligned} \tag{2.7}$$

A diagonal matrix D related to the double antipode action can be defined as

$$\begin{aligned}
&\kappa^2(M^I_{\ J}) = D^I_{\ K} M^K_{\ L} (D^{-1})^L_{\ J} \\
&D^0_{\ 0} = D^3_{\ 3} = 1 \qquad D^+_{\ +} = q^2 \qquad D^-_{\ -} = q^{-2}.
\end{aligned} \tag{2.8}$$

The linear functionals [22], $(L^\pm)^a_{\ b}$, defined by their values on the entries $T^a_{\ b}$, belong to the dual Hopf algebra $\mathcal{A}'$:

$$\begin{aligned}
&(L^+)^a_{\ b}\left(T^c_{\ d}\right) = q^{-1/2}(\hat{R}_q)^{ac}_{\ \ db} \qquad (L^-)^a_{\ b}\left(T^c_{\ d}\right) = q(\hat{R}_q^{-1})^{ac}_{\ \ db} \\
&\left(\hat{R}_q\right)^{ab}_{\ \ ef} (L^\pm)^s_{\ d} (L^\pm)^r_{\ c} = (L^\pm)^b_{\ s} (L^\pm)^a_{\ r} \left(\hat{R}_q\right)^{rs}_{\ \ cd} \\
&\left(\hat{R}_q\right)^{ab}_{\ \ ef} (L^+)^s_{\ d} (L^-)^r_{\ c} = (L^-)^b_{\ s} (L^+)^a_{\ r} \left(\hat{R}_q\right)^{rs}_{\ \ cd}.
\end{aligned} \tag{2.9}$$

The q-deformed exterior derivative δ is defined as a map from $\mathcal{A}$ to bimodule Γ:

$$\begin{aligned} &\delta : \mathcal{A} \to \Gamma \\ &\rho = \sum \alpha\delta\beta \qquad \text{if } \rho \in \Gamma \end{aligned} \tag{2.10}$$

where $\alpha,\ \beta \in \mathcal{A}$. When the operator δ acts on the fields, Watamura [15] called it the BRST transformation operator and the fields in Γ have the ghost number 1.

A left action $\Delta_{\rm L}$ and a right action $\Delta_{\rm R}$ of the quantum group on Γ are defined as follows:

$$\begin{aligned} &\Delta_{\rm L} : \Gamma \to \mathcal{A} \otimes \Gamma \qquad \Delta_{\rm L}(\alpha\delta\beta) = \Delta(\alpha)(id \otimes \delta)\Delta(\beta) \\ &\Delta_{\rm R} : \Gamma \to \Gamma \otimes \mathcal{A} \qquad \Delta_{\rm R}(\alpha\delta\beta) = \Delta(\alpha)(\delta \otimes id)\Delta(\beta). \end{aligned} \tag{2.11}$$

The bases [8] of the right-invariant elements of Γ are denoted by η^J, satisfying:

$$\begin{aligned} &\Delta_{\rm R}(\eta^J) = \eta^J \otimes 1 \qquad \Delta_{\rm L}(\eta^J) = M^J{}_K \otimes \eta^K \\ &\alpha\eta^J = \eta^K(\alpha * L^J_K) \qquad \alpha \in \mathcal{A} \qquad L^J_K \in \mathcal{A}' \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} &L_I{}^J = (\sigma^J)_a{}^d \kappa' \left((L^+)^a{}_b\right) (L^-)^c{}_d (\sigma_I)^b{}_c \\ &L_I{}^J(\alpha\beta) = L_I{}^K(\alpha) L_K{}^J(\beta) \qquad L_I{}^J(1) = \delta^J_I \\ &(\rho * L_I{}^J) = (L_I{}^J \otimes id)\Delta_{\rm L}(\rho). \end{aligned} \tag{2.13}$$

The basis of the left-invariant element of Γ is easy to calculate from η^J:

$$\omega^J = \kappa(M^J{}_K)\eta^K \qquad \Delta_{\rm L}(\omega^J) = 1 \otimes \omega^J \qquad \Delta_{\rm R}(\omega^J) = \omega^K \otimes \kappa(M^J{}_K). \tag{2.14}$$

As the analogue of the ordinary permutation operator, a bimodule automorphism Λ in $\Gamma \otimes \Gamma$ is defined by

$$\begin{aligned} &\Lambda(\omega^J \otimes \eta^K) = \eta^K \otimes \omega^J \\ &\Lambda(a\tau) = a\Lambda(\tau) \qquad \Lambda(\tau a) = \Lambda(\tau)a \qquad a \in \mathcal{A} \qquad \tau \in \Gamma \otimes \Gamma. \end{aligned} \tag{2.15}$$

Thus, we have

$$\Lambda(\eta^I \otimes \eta^J) = \Lambda^{IJ}{}_{KL}\eta^K \otimes \eta^L \qquad \Lambda^{IJ}{}_{KL} = L^J_K(M^I{}_L). \tag{2.16}$$

The non-vanishing components of $\Lambda^{IJ}{}_{KL}$ can be listed as follows

$$\begin{aligned} &\Lambda^{ij}{}_{k\ell} = (\Lambda^{-1})^{ij}{}_{k\ell} = \delta^i_k\delta^j_\ell + \bar{f}^{ij}_n\ f^n_{k\ell} \qquad \Lambda^{00}{}_{00} = (\Lambda^{-1})^{00}{}_{00} = 1 \\ &\Lambda^{i0}{}_{jk} = (\Lambda^{-1})^{0i}{}_{jk} = \lambda\ f^i_{jk} \qquad \Lambda^{jk}{}_{0i} = (\Lambda^{-1})^{jk}{}_{i0} = \lambda\ \bar{f}^{jk}_i \\ &\Lambda^{0i}{}_{j0} = (\Lambda^{-1})^{i0}{}_{0j} = \delta^i_j \qquad \Lambda^{i0}{}_{0j} = (\Lambda^{-1})^{0i}{}_{j0} = (\lambda^2 + 1)\ \delta^i_j \end{aligned} \tag{2.17}$$

where the non-vanishing components of $f^i_{jk} = -\bar{f}^{jk}_i$ are [15]

$$f^+_{+3} = f^-_{3-} = q \qquad f^+_{3+} = f^-_{-3} = -q^{-1} \qquad f^3_{-+} = -f^3_{+-} = 1 \qquad f^3_{33} = \lambda. \tag{2.18}$$

Watamura [15] introduced an operator for given tensors $X^{ab}{}_{cd}$ and $Y^{ab}{}_{cd}$:

$$(X, Y)^{IJ}{}_{KL} = (\sigma^I)_{a_1a_2}(\sigma^J)_{b_1b_2} X^{a_1e_1}{}_{c_1e_3} \left(\hat{R}_q^{-1}\right)^{a_2b_1}{}_{e_1e_2} Y^{e_2b_2}{}_{e_4d_2} \left(\hat{R}_q\right)^{e_3e_4}{}_{c_2d_1} (\sigma_K)^{c_1c_2}(\sigma_{\rm L})^{d_1d_2}. \tag{2.19}$$

Then, $\Lambda^{IJ}{}_{KL}$ can be expressed as a combination of four projection operators [15]:

$$\begin{aligned} \Lambda^{IJ}{}_{KL} = \left(\hat{R}^{-1}, \hat{R}\right)^{IJ}{}_{KL} &= (\mathcal{P}_S, \mathcal{P}_S)^{IJ}{}_{KL} - q^{-2}(\mathcal{P}_S, \mathcal{P}_A)^{IJ}{}_{KL} - q^2(\mathcal{P}_A, \mathcal{P}_S)^{IJ}{}_{KL} \\ &+ (\mathcal{P}_A, \mathcal{P}_A)^{IJ}{}_{KL} \end{aligned}$$

$$(\Lambda^{-1})^{IJ}{}_{KL} = \left(\hat{R}^{-1}, \hat{R}\right)^{IJ}{}_{KL} = (\mathcal{P}_S, \mathcal{P}_S)^{IJ}{}_{KL} - q^2 (\mathcal{P}_S, \mathcal{P}_A)^{IJ}{}_{KL} - q^{-2} (\mathcal{P}_A, \mathcal{P}_S)^{IJ}{}_{KL} + (\mathcal{P}_A, \mathcal{P}_A)^{IJ}{}_{KL} . \tag{2.20}$$

$\Lambda^{IJ}{}_{KL}$ satisfy the Yang–Baxter equation [3]:

$$\Lambda^{IJ}{}_{LM}\Lambda^{MK}{}_{NR}\Lambda^{LN}{}_{PQ} = \Lambda^{JK}{}_{LM}\Lambda^{IL}{}_{PN}\Lambda^{NM}{}_{QR}. \tag{2.21}$$

The eigenvalues of the Λ matrix are 1, $-q^2$ and $-q^{-2}$:

$$\left(\Lambda + q^2\right)\left(\Lambda + q^{-2}\right)(\Lambda - \mathbf{1}) = 0. \tag{2.22}$$

The exterior product of the elements in Γ is given as follows

$$\begin{aligned} &\rho \wedge \rho' \equiv \rho \otimes \rho' - \Lambda\left(\rho \otimes \rho'\right) \\ &\eta^I \wedge \eta^J = \left(\delta^I_K \delta^J_L - \Lambda^{IJ}{}_{KL}\right)\left(\eta^K \otimes \eta^L\right). \end{aligned} \tag{2.23}$$

$\eta^I \wedge \eta^J$ is annihilated by the projection operator $\mathcal{P}_{SS}$ owing to (2.22) and (2.23):

$$\begin{aligned} &\mathcal{P}_{SS} = (\mathcal{P}_S, \mathcal{P}_S)^{IJ}{}_{KL} + (\mathcal{P}_A, \mathcal{P}_A)^{IJ}{}_{KL} = [2]^{-2}\left\{\Lambda + \Lambda^{-1} + (\lambda^2 + 2)\mathbf{1}\right\} \\ &(\mathcal{P}_{SS})^{IJ}{}_{KL}\left(\eta^K \wedge \eta^L\right) = 0. \end{aligned} \tag{2.24}$$

The projection operator $\mathcal{P}_{SA}$ now can be expressed as follows

$$\mathcal{P}_{SA} = (\mathcal{P}_S, \mathcal{P}_A)^{IJ}{}_{KL} + (\mathcal{P}_A, \mathcal{P}_S)^{IJ}{}_{KL} = [2]^{-2}\left\{\mathbf{2} - \Lambda - \Lambda^{-1}\right\}. \tag{2.25}$$

It is interesting to note that there is a projection operator $\mathcal{P}_{\rm Adj}$ with only the adjoint components:

$$\left(\mathcal{P}_{\rm Adj}\right)^{ij}{}_{k\ell} = \frac{[2]^2}{2(\lambda^2+2)}\left(\mathcal{P}_{SA}\right)^{ij}{}_{k\ell} = -(\lambda^2+2)^{-1}\bar{f}^{ij}_t f^t_{k\ell} \tag{2.26}$$

$$\begin{aligned} &\left(\mathcal{P}_{\rm Adj}\right)^{ij}{}_{rs}\left(\mathcal{P}_{\rm Adj}\right)^{rs}{}_{k\ell} = \left(\mathcal{P}_{\rm Adj}\right)^{ij}{}_{k\ell} \\ &\left(\mathcal{P}_{\rm Adj}\right)^{ij}{}_{k\ell}\left(\eta^k \wedge \eta^\ell\right) = \eta^i \wedge \eta^j \\ &\left(\mathcal{P}_{\rm Adj}\right)^{ij}{}_{k\ell}\ f^t_{ij} = f^t_{k\ell} \qquad \left(\mathcal{P}_{\rm Adj}\right)^{ij}{}_{k\ell}\ \bar{f}^{k\ell}_t = \bar{f}^{k\ell}_t. \end{aligned} \tag{2.27}$$

The actions of δ on $\mathcal{A}$ and on Γ can be expressed as follows

$$\delta\alpha = \frac{\mathrm{i}g}{\lambda}(\eta^0\alpha - \alpha\eta^0) = \eta^J(\alpha * \chi_J) \tag{2.28}$$

$$\delta\eta^I = \{\mathrm{i}g/\lambda\}\left\{\eta^0 \wedge \eta^I + \eta^I \wedge \eta^0\right\} \tag{2.29}$$

where

$$\lambda = q - q^{-1} \tag{2.30}$$

and $\chi_J \in \mathcal{A}'$ are the q-analogues of the tangent vectors at the identity element of the group and $(\,*\chi_J)$ are the analogues of the right invariant vector fields [13]:

$$\chi_J = \frac{\mathrm{i}g}{\lambda}\left(\epsilon\ \delta^0_J - L_J{}^0\right). \tag{2.31}$$

The q-deformed structure constants $\mathbf{C}_{JK}{}^I$ are

$$\begin{aligned} &\mathbf{C}_{JK}{}^I = \chi_J\left(M^I{}_K\right) \\ &\mathbf{C}_{JK}{}^0 = \mathbf{C}_{J0}{}^K = 0 \qquad \mathbf{C}_{0k}{}^j = -\mathrm{i}g\lambda\delta^j_k \qquad \mathbf{C}_{jk}{}^i = -\mathrm{i}g f^i_{jk} \end{aligned} \tag{2.32}$$

where the f^i_{jk} were given in (2.18). The so-called q-deformed Cartan–Maurer equation comes from (2.17), (2.29) and (2.32):

$$\begin{aligned}\delta\eta^I &= \eta^J \otimes \left(\eta^I * \chi_J\right)\\ &= \mathbf{C}_{JK}{}^{I}\eta^J \otimes \eta^K\\ &= (\lambda^2+2)^{-1}\,\mathbf{C}_{jk}{}^{I}\left(\eta^j \wedge \eta^k\right).\end{aligned} \tag{2.33}$$

From the condition $\delta^2\alpha = 0$, the functionals χ_J span the 'q-deformed Lie algebra' [5, 13]:

$$\chi_I\chi_J - \Lambda^{KL}{}_{IJ}\chi_K\chi_L = \mathbf{C}_{IJ}{}^{K}\,\chi_K. \tag{2.34}$$

Thus, the q-deformed structure constants satisfy the q-deformed Jacobi identities [5, 13]:

$$\mathbf{C}_{IR}{}^{P}\mathbf{C}_{JS}{}^{R} - \Lambda^{KL}{}_{IJ}\mathbf{C}_{KR}{}^{P}\mathbf{C}_{LS}{}^{R} = \mathbf{C}_{IJ}{}^{R}\mathbf{C}_{RS}{}^{P}. \tag{2.35}$$

For the adjoint components we obtain from (2.17)

$$\left(\mathcal{P}_{\rm Adj}\right)^{k\ell}{}_{ij}\,\mathbf{C}_{kr}{}^{p}\mathbf{C}_{\ell s}{}^{r} = \frac{\lambda^2+1}{\lambda^2+2}\mathbf{C}_{ij}{}^{r}\mathbf{C}_{rs}{}^{p}. \tag{2.36}$$

Now, we sketch the q-deformed BRST algebra $\mathcal{B}$ constructed by Watamura [15]. It is the algebra of the gauge fields, the ghost fields, matter fields and their derivatives on one spacetime point. The ghost fields have the same properties as the bases η^I in Γ, but are functions of the spacetime. We will still use the symbols η^I to denote the ghost fields if it does not cause confusion. The ghost fields in the BRST algebra have the ghost number 1, but the degree of form 0. The gauge potentials A^J have the degree of form 1, but the ghost number 0. The matter fields have zero ghost number and zero degree of form. There are two operators in $\mathcal{B}$: the spacetime exterior derivative operator d and the BRST transformation operator δ, that is the q-deformed exterior derivative operator, but acting on fields. The operator δ increases the ghost number by one and the operator d increases the degree of form by one. Watamura studied the covariant commutation relations among the matter fields, ghost fields, gauge potentials and their derivatives consistently with these two operators as well as the *-operation, the antimultiplicative inner involution.

First, in order to describe these covariant commutation relations by unified formulae, we forget the matter field in what follows and only discuss four kinds of fields in the BRST algebra $\mathcal{B}$: η^J, $d\eta^J$, A^J and dA^J. We introduce an index n that is equal to the difference between the degree of form and the ghost number. The indices n for η^J, $d\eta^J$, A^J and dA^J are -1, 0, 1 and 2, respectively. Both δ and d operations satisfy the Leibniz rule in the graded sense for the index n and are nilpotent operators:

$$\begin{aligned}&\delta^2 = 0 \qquad d^2 = 0 \qquad d\,\delta + \delta d = 0\\ &d(XY) = (dX)Y + (-1)^{n_x}X(dY)\\ &\delta(XY) = (\delta X)Y + (-1)^{n_x}X(\delta Y)\end{aligned} \tag{2.37}$$

where n_x is the index of X. Both d and δ are covariant for the left and right actions. For any element $\rho \in \mathcal{B}$ they satisfy

$$\begin{aligned}&\Delta_{\rm L}(\delta\rho) = (id \otimes \delta)\Delta_{\rm L}(\rho) \qquad \Delta_{\rm L}(d\rho) = (id \otimes d)\Delta_{\rm L}(\rho)\\ &\Delta_{\rm R}(\delta\rho) = (\delta \otimes id)\Delta_{\rm R}(\rho) \qquad \Delta_{\rm R}(d\rho) = (d \otimes id)\Delta_{\rm R}(\rho).\end{aligned} \tag{2.38}$$

Second, A^I are assumed [15] to have following properties, similar to η^I. Hereafter we neglect the wedge sign $\wedge$ for simplicity.

$$\begin{aligned}&(\mathcal{P}_{SS})^{IJ}{}_{KL}\left(A^K A^L\right) = 0 \qquad \left(\mathcal{P}_{\rm Adj}\right)^{ij}{}_{k\ell}\left(A^k A^\ell\right) = A^i A^j\\ &\frac{\mathrm{i}g}{\lambda}\left(A^0 A^i + A^i A^0\right) = (\lambda^2+2)^{-1}\mathbf{C}_{jk}{}^{i}A^j A^k.\end{aligned} \tag{2.39}$$

From the consistent conditions, $d\eta^J$ and dA^J have to satisfy the other conditions

$$(\mathcal{P}_{SA})^{IJ}{}_{KL}\left(d\eta^K d\eta^L\right) = 0 \qquad (\mathcal{P}_{SA})^{IJ}{}_{KL}\left(dA^K dA^L\right) = 0. \tag{2.40}$$

Third, the gauge potential is introduced in the covariant derivative. The covariant condition of the covariant derivative in the BRST transformation requires

$$\delta A^I = d\eta^I + \frac{ig}{\lambda}\left(\eta^0 A^I + A^I \eta^0\right). \tag{2.41}$$

Fourth, discuss two different fields X^J and Y^K with indices n_x and n_y, respectively. For definiteness assume $n_x > n_y$. The consistent condition requires

$$(-1)^{n_x n_y} X^I Y^J = \Lambda^{IJ}{}_{KL} Y^K X^L. \tag{2.42}$$

From (2.17) we have

$$\begin{aligned} &(-1)^{n_x n_y} X^0 Y^J = Y^J X^0 \\ &\{ig/\lambda\}\ \left(Y^0 X^I - (-1)^{n_x n_y}\ X^I Y^0\right) = \mathbf{C}_{JK}^{\ I} Y^J X^K. \end{aligned} \tag{2.43}$$

At last, the gauge fields F^J satisfy

$$\begin{aligned} &F^J = dA^J + ig\lambda^{-1}\left(A^0 A^J + A^J A^0\right) \\ &F^0 = dA^0 \qquad F^i = dA^i + (\lambda^2+2)^{-1}\mathbf{C}_{jk}{}^i A^j A^k \\ &F^I \eta^J = \eta^K F^L \Lambda^{IJ}{}_{KL} \end{aligned} \tag{2.44}$$

$$\begin{aligned} &\delta F^I = ig\lambda^{-1}\left(\eta^0\ F^I - F^I \eta^0\right) = \mathbf{C}_{JK}^{\ I} \eta^J F^K \\ &\delta F^0 = 0 \qquad \delta F^i = -ig\lambda\eta^0\ F^i + \mathbf{C}_{jk}{}^i \eta^j F^k \end{aligned} \tag{2.45}$$

$$\begin{aligned} dF^I &= -ig\lambda^{-1}\left(A^0 F^I - F^I A^0\right) \\ &= -ig\lambda^{-1}\left(A^0 dA^I - dA^I A^0\right) \\ &= -\mathbf{C}_{JK}^{\ I} A^J\ dA^K \\ dF^0 &= 0 \qquad dF^i = ig\lambda A^0 dA^i - \mathbf{C}_{jk}{}^i A^j dA^k. \end{aligned} \tag{2.46}$$

Please refer to the original paper [15] for the consistency of the q-deformed BRST algebra.

3. q-deformed Chern class

It is easy to understand that the second q-deformed Chern class has the following form [5, 14]:

$$P \equiv F^I F^J g_{IJ} \tag{3.1}$$

where we omit the possible constant factor in P. The q-deformed Killing form g_{IJ} is chosen from the condition

$$\delta P = 0 \qquad dP = 0. \tag{3.2}$$

In addition to (3.2) the difference in the q-deformed Chern class for the infinitesimal transformation of the gauge potential should be a total derivative. It will be proved in section 4 (see (4.2)) that the last condition is satisfied.

From (2.44) and (2.45) we have

$$\begin{aligned} \delta P &= \left\{\delta F^R\ F^S + F^R \delta F^S\right\} g_{RS} \\ &= ig\lambda^{-1}\left\{\eta^0 F^R F^S - F^R F^S \eta^0\right\} g_{RS} \\ &= ig\lambda^{-1}\eta^I F^J F^K \left\{\delta_I^0 g_{JK} - \Lambda^{S0}{}_{TK}\Lambda^{RT}{}_{IJ} g_{RS}\right\} \end{aligned}$$

namely, g_{IJ} has to satisfy

$$\delta^0_I g_{JK} = \Lambda^{S0}{}_{TK}\Lambda^{RT}{}_{IJ} g_{RS}. \tag{3.3}$$

It can be checked that (3.3) is equivalent to the similar conditions proposed by Bernard ((3.19) in [5]) and by Castellani ((30) in the second paper of [14]). We are going to show that the condition $dP = 0$ also holds if g_{IJ} satisfies (3.3). From (2.39) a product of four adjoint components of A^j must be vanishing, so that:

$$P = dA^R dA^S g_{RS} + \mathrm{i}g\lambda^{-1}\left\{dA^R(A^0A^S + A^SA^0) + (A^0A^R + A^RA^0)\,dA^S\right\} g_{RS}$$

$$\begin{aligned} dP &= \mathrm{i}g\lambda^{-1}\left\{dA^R(dA^0A^S - A^0dA^S + dA^SA^0 - A^SdA^0\right.\\ &\qquad \left. + (dA^0A^R - A^0dA^R + dA^RA^0 - A^RdA^0)dA^S\right\} g_{RS}\\ &= -\mathrm{i}g\lambda^{-1}\left(A^0dA^RdA^S - dA^RdA^SA^0\right) g_{RS}\\ &= -\mathrm{i}g\lambda^{-1}A^IdA^JdA^K\left\{\delta^0_I g_{JK} - \Lambda^{S0}{}_{TK}\Lambda^{RT}{}_{IJ} g_{RS}\right\}. \end{aligned}$$

From (2.17) we can solve (3.3) as follows.

(i) When $J = 0$, $I = i \neq 0$ and $K = k \neq 0$, we have

$$g_{0\ell} = 0. \tag{3.4a}$$

(ii) When $K = 0$, $I = i \neq 0$ and $J = j \neq 0$, we have:

$$g_{\ell 0} = 0. \tag{3.4b}$$

(iii) When $I = \pm$ and $J = K = 3$, we have:

$$g_{\pm 3} + g_{3\pm} = 0.$$

However, when $J = \pm$ and $I = K = 3$, we have

$$\mp q^{\mp 3} g_{\pm 3} + \lambda g_{3\pm} = 0.$$

Thus, we obtain

$$g_{\pm 3} = g_{3\pm} = 0. \tag{3.4c}$$

(iv) When $I = J = \pm$ and $K = 3$, we have

$$g_{++} = g_{--} = 0. \tag{3.4d}$$

(v) When $I = -J = \pm$ and $K = 3$, we have

$$g_{+-} = q g_{33} \qquad g_{-+} = q^{-1} g_{33}. \tag{3.4e}$$

For the rest of the cases there is no new restriction on g_{IJ}. It is worth noticing that the singlet and adjoint components of g_{IJ} are separated from each other, although these components are mixed in the commutative relations in $\mathcal{B}$. Furthermore, if one requires the q-deformed Chern class to contain only the adjoint components of gauge fields, the q-deformed Killing form as well as the q-deformed Chern class are determined uniquely up to a common factor.

We are going to choose the q-deformed Killing form g_{IJ} such that g_{IJ} satisfies (3.4) and is the analogue of the classical Killing form. The following q-deformed Killing form satisfies these conditions:

$$g_{IJ} = D^R{}_S \mathbf{C}_{IT}{}^S \mathbf{C}_{JR}{}^T = D^r{}_s \mathbf{C}_{It}{}^s \mathbf{C}_{Jr}{}^t \tag{3.5}$$

where the summed indices cannot equal to zero owing to (2.32). It is acceptable that the q-deformed Killing form is related to the double antipode action. From (2.32) the non-vanishing components of g_{IJ} are:

$$g_{00} = -g^2\lambda^2\ [3] \qquad q^{-1}g_{+-} = q g_{-+} = g_{33} = -g^2(\lambda^2 + 2). \tag{3.6}$$

Obviously, both g_{IJ} and g_{ij} (without 0 subscript) satisfy (3.4). From the q-deformed Killing form we may define two kinds of second q-deformed Chern class:

$$\begin{aligned} P &\equiv (F, F) = F^i F^j g_{ij} \\ &= -g^2(\lambda^2+2)\left(F^3F^3 + qF^+F^- + q^{-1}F^-F^+\right) \\ \hat{P} &\equiv F^I F^J g_{IJ} - P = -g^2\lambda^2[3]F^0F^0 \end{aligned} \tag{3.7}$$

where P and $\hat{P}$ contain only the adjoint components and the singlet of F^j, respectively. From now on we will mainly discuss q-deformed Chern class P. A similar calculation for $\hat{P}$ is much simpler than that for P. From a similar condition, Castellani [14] found another q-deformed Killing form with an additional parameter r, that may describe the mixture of two kinds of component.

4. q-deformed Chern–Simons

In the classical case Zumino [18, 19] introduced a homotopy operator k to compute the Chern–Simons. Now we generalize his method to compute the q-deformed Chern–Simons.

Introduce a q-deformed homotopy operator k that is nilpotent and satisfies the Leibniz rule in the graded sense for the index n:

$$k^2 = 0 \qquad dk + kd = \mathbf{1}. \tag{4.1}$$

If k exists, we can compute the q-deformed Chern–Simons $Q(A)$ from the q-deformed Chern class

$$\begin{aligned} &P = (dk+kd)P = d\,(kP) = dQ(A) \\ &Q(A) = kP \end{aligned} \tag{4.2}$$

where we have used (3.2).

Introduce a real parameter t, $0 \leqslant t \leqslant 1$. When t changes from 0 to 1, the gauge potentials A_t^J change from 0 to A^J:

$$\begin{aligned} A_t^J &= tA^J \\ F_t^J &= tdA^J + \frac{igt^2}{\lambda}\left(A^0A^J + A^JA^0\right) \\ &= tF^J + \frac{ig(t^2-t)}{\lambda}\left(A^0A^J + A^JA^0\right). \end{aligned} \tag{4.3}$$

Owing to our definition (2.6) of the q-number we have to change the usual definition [24] slightly for the q-deformed derivative and the q-deformed integral. Define the q-deformed derivative along t by

$$\frac{\partial}{\partial_q t} f(t) = \frac{f(qt) - f(q^{-1}t)}{t(q-q^{-1})} \tag{4.4}$$

satisfying the q-deformed Leibniz rule:

$$\frac{\partial}{\partial_q t} f(t)g(t) = \frac{\partial f(t)}{\partial_q t} g(qt) + f(q^{-1}t)\frac{\partial g(t)}{\partial_q t}. \tag{4.5}$$

The q-deformed integral is defined by

$$\int_0^{t_0} d_q t f(t) = t_0(1-q^2)\sum_{k=0}^{\infty} q^{2k} f(q^{2k+1}t_0). \tag{4.6}$$

At least for a polynomial, the q-deformed integral is the inverse of the q-deformed derivative. For example,

$$\frac{\partial}{\partial_q t} t^m = [m] t^{m-1} \qquad \int_0^{t_0} d_q t t^{m-1} = t_0^m/[m].$$

Now, define the q-deformed Lie derivative $\hat{\delta}_q$ along t in the gauge space:

$$\hat{\delta}_q \equiv d_q t \frac{\partial}{\partial_q t} \tag{4.7}$$

and the q-deformed operator ℓ_t that satisfies the q-deformed Leibniz rule in the graded sense for the index n:

$$\begin{aligned} &\ell_t A_t^J = 0 \qquad \ell_t F_t^J = \hat{\delta}_q A_t^J = d_q t A^J \\ &\ell_t \{f(t) g(t)\} = \{\ell_t f(t)\} g(qt) + (-1)^n f(q^{-1}t)\{\ell_t g(t)\} \end{aligned} \tag{4.8}$$

where f has index n.

It is easy to check that for all formal polynomials (vanishing at $F_t^J = 0$ and $A_t^J = 0$) we have

$$\begin{aligned} &\ell_t \ell_t = 0 \\ &\ell_t d + d\ \ell_t = \hat{\delta}_q = d_q t \frac{\partial}{\partial_q t} \\ &\hat{\delta}_q d = d \hat{\delta}_q \qquad \hat{\delta}_q \ell_t = \ell_t \hat{\delta}_q. \end{aligned} \tag{4.9}$$

Comparing this with (4.1) we obtain

$$k = \int_0^1 \ell_t. \tag{4.10}$$

Now, we are able to compute the q-deformed Chern–Simons by (4.2):

$$\begin{aligned} \ell_t P_t &= \langle \ell_t F_t, F_{qt} \rangle + \langle F_{q^{-1}t}, \ell_t F_t \rangle \\ &= d_q t \left\{ \langle A, F_{qt} \rangle + \langle F_{q^{-1}t}, A \rangle \right\} \\ &= d_q t \left\{ t[2] \langle A, dA \rangle + \mathrm{i} g \lambda^{-1} t^2 (\lambda^2 + 2) \langle A, (A^0 A + A A^0) \rangle \right\} \\ &= d_q t \left\{ t[2] \langle A, dA \rangle + \frac{t^2(\lambda^2+2)}{\lambda^2+1} \langle A, A, A \rangle \right\} \end{aligned} \tag{4.11}$$

where we have used the definition (A.1) and relation (A.5).

$$\begin{aligned} Q(A) &= k\ P_t \\ &= \langle A, dA \rangle + \{[4]/[6]\} \langle A, A, A \rangle \\ &= \langle A, F \rangle - \{[2]/[6]\} \langle A, A, A \rangle. \end{aligned} \tag{4.12}$$

Similarly, we can compute $\hat{Q}(A)$ from $\hat{P} = d\hat{Q}(A)$:

$$\hat{Q}(A) = k \hat{P}_t = -g^2 \lambda^2 [3] A^0 d A^0. \tag{4.13}$$

It is easy to prove by direct calculation that

$$dQ(A) = P \qquad d\hat{Q}(A) = \hat{P}. \tag{4.14}$$

In fact, the second equation is obvious and the first one can be proved in terms of the formulae given in the appendix:

$$\begin{aligned}
dQ(A) &= \langle dA, dA\rangle + \{[4]/[6]\}\{\langle dA, A, A\rangle - \langle A, dA, A\rangle + \langle A, A, dA\rangle\} \\
&= \langle dA, dA\rangle + (\lambda^2+1)^{-1}\{\langle dA, A, A\rangle + \langle A, A, dA\rangle\} \\
&= \langle dA, dA\rangle + \frac{\mathrm{i}g}{\lambda}\left\{\langle dA, (A^0A + AA^0)\rangle + \langle (A^0A + AA^0), dA\rangle\right\} \\
&= \langle F, F\rangle = P.
\end{aligned}$$

We find that the components of the identity and the adjoint representations are separated in the q-deformed Chern class and in the q-deformed Chern–Simons, although they are mixed in the commutative relations of BRST algebra.

5. q-deformed cocycle hierarchy

The gauge fields F^J, just like those in the classical case [24], are invariant under the transformation

$$A^J \to A^J - \eta^J \qquad d \to d + \delta. \tag{5.1}$$

In fact,

$$F^J \to \mathcal{F}^J = (d+\delta)(A^J - \eta^J) + \frac{\mathrm{i}g}{\lambda}\left\{(A^0 - \eta^0)(A^J - \eta^J) + (A^J - \eta^J)(A^0 - \eta^0)\right\}$$

$$\begin{aligned}
F^J &+ \left\{\delta A^J - d\eta^J - \frac{\mathrm{i}g}{\lambda}(\eta^0 A^J + A^J\eta^0)\right\} - \left\{\delta\eta^J - \frac{\mathrm{i}g}{\lambda}\ (\eta^0\eta^J + \eta^J\eta^0 - A^0\eta^J - \eta^J A^0)\right\} \\
&= F^J.
\end{aligned}$$

Now, transforming (4.14) and expanding it by the ghost number, we obtain

$$\begin{aligned}
Q(A-\eta) &= \langle A-\eta, F\rangle - \{[2]/[6]\}\langle A-\eta, A-\eta, A-\eta\rangle \\
&= \omega_3^0 + \omega_2^1 + \omega_1^2 + \omega_0^3
\end{aligned}$$

$$\begin{aligned}
P &= (d+\delta)\ Q(A-\eta) \\
&= d\omega_3^0 + \left\{\delta\omega_3^0 + d\omega_2^1\right\} + \left\{\delta\omega_2^1 + d\omega_1^2\right\} + \left\{\delta\omega_1^2 + d\omega_0^3\right\} + \delta\omega_0^3
\end{aligned} \tag{5.2}$$

where the subscripts denote the degrees of form of the quantities and the superscripts denote the ghost numbers. In the two sides of (5.2) the quantities with the same degree of form and the same ghost number should be equal to each other, respectively:

$$\begin{aligned}
&P = d\omega_3^0 \qquad \delta\omega_3^0 + d\omega_2^1 = 0 \qquad \delta\omega_2^1 + d\omega_1^2 = 0 \\
&\delta\omega_1^2 + d\omega_0^3 = 0 \qquad \delta\omega_0^3 = 0.
\end{aligned} \tag{5.3}$$

We are going to derive the explicit forms of these ω and simplify them by the formulae given in the appendix.

$$\omega_3^0 = Q(A) = \langle A, F\rangle - \{[2]/[6]\}\langle A, A, A\rangle = \langle A, dA\rangle + \{[4]/[6]\}\langle A, A, A\rangle \tag{5.4}$$

$$\begin{aligned}
\omega_2^1 &= -\langle \eta, dA\rangle - \{\mathrm{i}g/\lambda\}\langle \eta, (A^0A + AA^0)\rangle + \{[2]/[6]\}\{\langle \eta, A, A\rangle + \langle A, \eta, A\rangle + \langle A, A, \eta\rangle\} \\
&= -\langle \eta, dA\rangle
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
\omega_1^2 &= -\{[2]/[6]\}\{\langle \eta, \eta, A\rangle + \langle \eta, A, \eta\rangle + \langle A, \eta, \eta\rangle\} \\
&= -(\lambda^2+1)^{-1}\langle \eta, A, \eta\rangle
\end{aligned} \tag{5.6}$$

$$\omega_0^3 = \{[2]/[6]\}\langle \eta, \eta, \eta \rangle. \tag{5.7}$$

Similarly, we also have:

$$\begin{aligned}
&\hat{Q}(A-\eta) = -g^2\lambda^2[3]\left(A^0-\eta^0\right)\ F^0 = \hat{\omega}_3^0 + \hat{\omega}_2^1 \\
&\hat{\omega}_3^0 = \hat{Q}(A) = -g^2\lambda^2[3]A^0F^0 \\
&\hat{\omega}_2^1 = g^2\lambda^2[3]\eta^0F^0 \\
&\hat{P} = d\hat{\omega}_3^0 \qquad \delta\hat{\omega}_3^0 + d\hat{\omega}_2^1 = 0 \qquad \delta\hat{\omega}_2^1 = 0.
\end{aligned} \tag{5.8}$$

Equation (5.8) is obvious. We are going to prove (5.3) again directly by the formulae in the appendix.

$$\begin{aligned}
\delta\omega_3^0 + d\omega_2^1 &= \langle \delta A, dA\rangle + \langle A, d(\delta A)\rangle \\
&\quad + \{[4]/[6]\}\{\langle \delta A, A, A\rangle - \langle A, \delta A, A\rangle + \langle A, A, \delta A\rangle\} \\
&\quad - \langle d\eta, dA\rangle \\
&= \{ig/\lambda\}\left\{\langle(\eta^0A + A\eta^0), dA\rangle + \langle A, (d\eta^0A - \eta^0dA + dA\eta^0 - Ad\eta^0)\rangle\right\} \\
&\quad + \{[4]/[6]\}\{\langle d\eta, A, A\rangle - \langle A, d\eta, A\rangle + \langle A, A, d\eta\rangle\} \\
&\quad + \frac{ig[4]}{\lambda[6]}\left\{\langle(\eta^0A + A\eta^0), A, A\rangle - \langle A, (\eta^0A + A\eta^0), A\rangle\right. \\
&\quad \left. + \langle A, A, (\eta^0A + A\eta^0)\rangle\right\} \\
&= \{ig/\lambda\}\left\{\eta^0\langle A, dA\rangle + \langle Ad\eta^0, A\rangle + \langle A, dA\rangle\ \eta^0\right\} \\
&\quad + \{[4][3]/[6]\}\langle d\eta, A, A\rangle + \frac{ig[4]}{\lambda[6]}\left\{\eta^0\langle A, A, A\rangle + \langle A, A, A\rangle\eta^0\right\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\delta\omega_2^1 + d\omega_1^2 &= -\langle \delta\eta, dA\rangle - \{ig/\lambda\}\langle \eta, (d\eta^0A - \eta^0dA + dA\eta^0 - Ad\eta^0)\rangle \\
&\quad - (\lambda^2+1)^{-1}\{\langle d\eta, A, \eta\rangle - \langle \eta, dA, \eta\rangle + \langle \eta, A, d\eta\rangle\} \\
&= -(\lambda^2+1)^{-1}\langle \eta, \eta, dA\rangle - \{ig/\lambda\}\langle \eta d\eta^0, A\rangle + ig\lambda\ \eta^0\langle \eta, dA\rangle \\
&\quad - \lambda^2(\lambda^2+1)^{-1}\langle \eta, \eta, dA\rangle + \frac{\lambda^2+2}{\lambda^2+1}\langle \eta, \eta, dA\rangle \\
&\quad + \{ig/\lambda\}(\lambda^2+1)\langle \eta d\eta^0, A\rangle - \frac{\lambda^2+2}{\lambda^2+1}\langle \eta, d\eta, A\rangle \\
&\quad - (\lambda^2+1)^{-1}\left\{-\langle \eta, d\eta, A\rangle + ig\lambda(\lambda^2+1)\eta^0\langle \eta, dA\rangle\right. \\
&\quad \left. + \langle \eta, \eta, dA\rangle + ig\lambda(\lambda^2+1)\langle \eta d\eta^0, A\rangle - (\lambda^2+1)\langle \eta, d\eta, A\rangle\right\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\delta\omega_1^2 + d\omega_0^3 &= -(\lambda^2+1)^{-1}\{\langle \delta\eta, A, \eta\rangle - \langle \eta, \delta A, \eta\rangle + \langle \eta, A, \delta\eta\rangle\} \\
&\quad + \{[2]/[6]\}\{\langle d\eta, \eta, \eta\rangle - \langle \eta, d\eta, \eta\rangle + \langle \eta, \eta, d\eta\rangle\} \\
&= -(\lambda^2+1)^{-1}\left\{(\lambda^2+1)^{-1}\langle \eta, \eta, A, \eta\rangle - \langle \eta, d\eta, \eta\rangle\right. \\
&\quad \left. - \{ig/\lambda\}\langle \eta, (\eta^0A + A\eta^0), \eta\rangle + (\lambda^2+1)^{-1}\langle \eta, A, \eta, \eta\rangle\right\} \\
&\quad - \{[2][3]/[6]\}\langle \eta, d\eta, \eta\rangle \\
&= 0.
\end{aligned}$$

A product of four adjoint components of η^j must be vanishing, so we have

$$\delta\omega_0^3 = \{[2]/[6]\}\{\langle \delta\eta, \eta, \eta\rangle - \langle \eta, \delta\eta, \eta\rangle + \langle \eta, \eta, \delta\eta\rangle\}$$

$$= \{[3][2]^2/[6]^2\}\langle \eta, \eta, \eta, \eta \rangle$$
$$= 0.$$

6. Conclusions and discussions

Based on the q-deformed BRST algebra, we have studied the q-deformed Killing form and the second q-deformed Chern class for the quantum group $SU_q(2)$ from the q-gauge covariant condition. We find that, although the components of the identity and the adjoint representations have to be mixed in the commutative relations of the q-deformed BRST algebra, the q-deformed Killing form and the second Chern class can contain only one kind of component—a singlet or adjoint component. We compute the second q-deformed Chern class that contains only the adjoint components in detail. Introducing a q-deformed homotopy operator, that is the quantum analogue of the homotopy operator presented by Chern and Zumino [18, 19], we are able to compute the q-deformed Chern–Simons. Then, the q-deformed cocycle hierarchy can be calculated by the invariance of P in the transformation (5.1). These method can be generalized to compute the higher q-deformed Chern classes and Chern–Simons and to the quantum groups $SU_q(N)$.

Besides these generalizations, there are some interesting problems left to be studied. Although the framework of quantum bundles has been given [11], the relations between the q-deformed gauge transformation parameter and the q-deformed BRST transformation need to be further studied. Here we have concentrated on the algebraic and homotopy formalism. We are able to discuss the double cohomology and cocycle hierarchy in terms of two operators d and δ. However, we have not paid enough attention to the deep geometrical meaning of the q-deformed Chern characters. An important problem is how to integrate the q-deformed Chern class to obtain the q-deformed characteristic number. We have not studied the physical application of the q-deformed Chern characters, for example how to establish the q-deformed Yang–Mills theory, the quantization of the q-deformed field theory and the meaning of the q-deformed Faddeev–Popov ghost and q-anomaly, etc. We will consider these problems in future work.

Acknowledgments

This work was supported by the National Natural Science Foundation of China and Grant No. LWTZ-1298 of the Chinese Academy of Sciences.

Appendix A. Some recursive relations

Generalizing (3.1) we define

$$\langle X_1, X_2, \cdots, X_m \rangle = X_1^{j_1} X_2^{j_2} \cdots X_m^{j_m} g_{j_1 j_2 \cdots j_m} \tag{A.1}$$

where

$$g_{j_1 j_2 \cdots j_n} \equiv D_{t_1}^{t_0} \mathbf{C}_{j_1 t_2}^{\ \ t_1} \mathbf{C}_{j_2 t_3}^{\ \ t_2} \cdots \mathbf{C}_{j_m t_0}^{\ \ t_m}. \tag{A.2}$$

X_n are fields η^J, $d\eta^J$, A^J or dA^J in the BRST algebra $\mathcal{B}$. They can also be replaced by, for example, XY^0, Y^0X or F. The following relations are easy to prove by direct calculation:

$$\begin{aligned} &\Lambda^{RS}{}_{IJ}\, g_{RS} = g_{IJ} &\qquad& \Lambda^{rs}{}_{ij}\, g_{rs} = g_{ij} \\ &f^i_{rs}\, g_{rs} = 0 && \left(\mathcal{P}_{\rm Adj}\right)^{rs}{}_{ij}\, g_{rs} = 0 \\ &\left(\mathcal{P}_{\rm Adj}\right)^{rs}{}_{ij}\, g_{rsk} = g_{ijk} && \left(\mathcal{P}_{\rm Adj}\right)^{rs}{}_{jk}\, g_{irs} = g_{ijk}. \end{aligned} \tag{A.3}$$

Now, we are going to derive four important recursive relations. Assume X^J and Y^J are two different fields in $\mathcal{B}$ with the indices $n_x > n_y$. Z^J denotes a field in $\mathcal{B}$, or a field multiplied by the zero component of this or another field. Since the following relations are linear for the field Z^J, Z^J can also be replaced by their linear combination, for example, F^J.

From (2.42) and (2.17) we have

$$\begin{aligned}(-1)^{n_x n_y} X^i Y^0 &= Y^J X^K \Lambda^{i0}{}_{JK} \\ &= (\lambda^2+1) Y^0 X^i - (ig)^{-1} \lambda Y^j X^k\ \mathbf{C}_{jk}{}^i.\end{aligned}$$

As a result of (2.36),

$$(-1)^{n_x n_y} X^i Y^0 \mathbf{C}_{is}{}^r = (\lambda^2+1) Y^0 X^i \mathbf{C}_{is}{}^r - \frac{\lambda(\lambda^2+2)}{ig(\lambda^2+1)} Y^u X^v \left(\mathcal{P}_{\rm Adj}\right)^{jk}{}_{uv}\ \mathbf{C}_{jt}{}^r \mathbf{C}_{ks}{}^t.$$

Now, according to the definitions (A.1) and (A.3), we have

$$\begin{aligned}(-1)^{n_x n_y} \langle Z, X\rangle Y^0 &= (\lambda^2+1)\langle Z, Y^0 X\rangle - \frac{\lambda(\lambda^2+2)}{ig(\lambda^2+1)}\langle Z, Y, X\rangle \\ (-1)^{n_x n_y} \langle X Y^0, Z\rangle &= (\lambda^2+1) Y^0 \langle X, Z\rangle - \frac{\lambda(\lambda^2+2)}{ig(\lambda^2+1)}\langle Y, X, Z\rangle.\end{aligned} \tag{A.4}$$

Similarly, from (2.36) and (2.39) we have:

$$\begin{aligned}\{ig/\lambda\}\left(A^0 A^i + A^i A^0\right) \mathbf{C}_{is}{}^r &= (\lambda^2+2)^{-1} A^j A^k \mathbf{C}_{jk}{}^i \mathbf{C}_{is}{}^r \\ &= (\lambda^2+1)^{-1} A^j A^k \mathbf{C}_{jt}{}^r \mathbf{C}_{ks}{}^t.\end{aligned}$$

The equation holds for η^J, too. Thus, we have

$$\begin{aligned}&\{ig/\lambda\}\langle\cdots, Z_1, (A^0 A + A A^0), Z_2, \cdots\rangle = (\lambda^2+1)^{-1}\langle\cdots, Z_1, A, A, Z_2, \cdots\rangle \\ &\{ig/\lambda\}\langle\cdots, Z_1, (\eta^0\eta + \eta\eta^0), Z_2, \cdots\rangle = (\lambda^2+1)^{-1}\langle\cdots, Z_1, \eta, \eta, Z_2, \cdots\rangle.\end{aligned} \tag{A.5}$$

From (2.35) and (2.43) we obtain

$$\begin{aligned}(-1)^{n_x n_y} X^I Y^J \mathbf{C}_{It}{}^r \mathbf{C}_{Js}{}^t &= Y^K X^L \Lambda^{IJ}{}_{KL} \mathbf{C}_{It}{}^r \mathbf{C}_{Js}{}^t \\ &= Y^K X^L \left\{\mathbf{C}_{Kt}{}^r \mathbf{C}_{Ls}{}^t - \mathbf{C}_{KL}{}^t \mathbf{C}_{ts}{}^r\right\} \\ &= Y^K X^L \mathbf{C}_{Kt}{}^r \mathbf{C}_{Ls}{}^t - \{ig/\lambda\}\left(Y^0 X^j - (-1)^{n_x n_y} X^j Y^0\right) \mathbf{C}_{js}{}^r.\end{aligned}$$

Then,

$$\begin{aligned}&\left\{(-1)^{n_x n_y} X^i Y^j - Y^i X^j\right\} \mathbf{C}_{it}{}^r \mathbf{C}_{js}{}^t \\ &\quad= \left\{(-1)^{n_x n_y} X^I Y^J - Y^I X^J\right\} \mathbf{C}_{It}{}^r \mathbf{C}_{Js}{}^t \\ &\quad\quad + ig\lambda\left\{(-1)^{n_x n_y} X^j Y^0 - Y^0 X^j\right\} \mathbf{C}_{js}{}^r \\ &\quad= -ig\lambda^{-1}(\lambda^2+1)\left\{Y^0 X^j - (-1)^{n_x n_y} X^j Y^0\right\} \mathbf{C}_{js}{}^r\end{aligned}$$

So we have

$$\begin{aligned}(-1)^{n_x n_y}\langle\cdots, Z_1, X, Y, Z_2, \cdots\rangle &= \langle\cdots, Z_1, Y, X, Z_2, \cdots\rangle \\ &\quad -ig\lambda^{-1}(\lambda^2+1)\langle\cdots, Z_1, (Y^0 X - (-1)^{n_x n_y} X Y^0), Z_2, \cdots\rangle.\end{aligned} \tag{A.6}$$

By making use of (A.4) we have

$$\begin{aligned}(-1)^{n_x n_y}\langle Z, X, Y\rangle &= ig\lambda(\lambda^2+1)\langle Z, Y^0 X\rangle - (\lambda^2+1)\ \langle Z, Y, X\rangle \\ (-1)^{n_x n_y}\langle X, Y, Z\rangle &= ig\lambda(\lambda^2+1)\langle Y^0 X, Z\rangle - (\lambda^2+1)\langle Y, X, Z\rangle.\end{aligned} \tag{A.7}$$

From these four recursive relations and (A.3), we can derive the following useful formulae. First of all, from (A.3) we have

$$\langle \eta, \eta \rangle = \langle A, A \rangle = 0 \qquad (-1)^{n_x n_y} \langle X, Y \rangle = \langle Y, X \rangle. \tag{A.8}$$

From (A.5) we have

$$\begin{aligned} \langle F, A \rangle = \langle A, F \rangle &= \langle A, dA \rangle + \frac{\mathrm{i}g}{\lambda} \langle A, (A^0 A + A A^0) \rangle \\ &= \langle A, dA \rangle + (\lambda^2 + 1)^{-1} \langle A, A, A \rangle. \end{aligned} \tag{A.9}$$

If the fields Y^J and Z^J are not the field η^J, we have

$$\langle Y, Z \rangle \eta^0 = (-1)^{n_y + n_z} \eta^0 \langle Y, Z \rangle. \tag{A.10}$$

Let Z^J be a field in $\mathcal{B}$ with the index n_z and let X^J denote the field η^J or A^J with the index $n_x = -1$ or 1, respectively. From the recursive relations we obtain that, if $n_z < n_x$,

$$\begin{aligned} &\langle X, X, Z \rangle = \langle Z, X, X \rangle \\ &(-1)^{n_x n_z} \langle X, Z, X \rangle = -(\lambda^2 + 1) \langle Z, X, X \rangle \\ &\langle X, X, Z \rangle - (-1)^{n_x n_z} \langle X, Z, X \rangle + \langle Z, X, X \rangle = [3] \langle Z, X, X \rangle. \end{aligned} \tag{A.11}$$

If $n_z > n_x$, we have

$$\begin{aligned} &\langle Z, X, X \rangle = \mathrm{i}g\lambda(\lambda^2 + 1)(\lambda^2 + 2) X^0 \langle X, Z \rangle + (\lambda^2 + 1) \langle X, X, Z \rangle \\ &(-1)^{n_x n_z} \langle X, Z, X \rangle = -\mathrm{i}g\lambda(\lambda^2 + 1) X^0 \langle X, Z \rangle - \langle X, X, Z \rangle \\ &\qquad = -(\lambda^2 + 2)^{-1} \{ \langle X, X, Z \rangle + \langle Z, X, X \rangle \} \\ &\langle X, X, Z \rangle - (-1)^{n_x n_z} \langle X, Z, X \rangle + \langle Z, X, X \rangle = -(-1)^{n_x n_z} [3] \langle X, Z, X \rangle. \end{aligned} \tag{A.12}$$

At last, through direct calculation we have

$$\langle A, A, A \rangle\, \eta^0 = -\,\eta^0\, \langle A, A, A \rangle = \mathrm{i}g^3 (\lambda^2 + 1)(\lambda^2 + 2)\, \eta^0 A^3 A^+ A^-. \tag{A.13}$$

References

[1] Manin Yu I 1988 Quantum groups and non-commutative geometry *Preprint* Montreal University, CRM-1561

[2] Connes A 1986 *Publ. Math. IHES* **62** 41

[3] Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613; 1989 **122** 125

[4] Wess J and Zumino B 1990 *Nucl. Phys. (Proc. Suppl.)* B **18** 302

[5] Bernard D 1990 Quantum Lie algebras and differential calculus on quantum groups (Proc. 1990 Yukawa Int. Seminar (Kyoto)) *Phys. Lett.* **260B** (1991) 389

[6] Jurčo B 1991 *Lett. Math. Phys.* **22** 177

[7] Zumino B 1991 Introduction to the differential geometry of quantum groups, LBL-31432 and UCB-PTH-62-91, notes of a plenary talk given at the 10th IAMP Conf., Leipzig

[8] Carow-Watamura U, Schlieker M, Watamura S and Weich W 1991 *Commun. Math. Phys.* **142** 605

[9] Isaev A P and Popowicz Z 1992 *Phys. Lett.* **281B** 271; Wu K and Zhang R J 1992 *Commun. Theor. Phys.* **17** 175

[10] Aref'eva I Ya and Volovich I V 1991 *Phys. Lett.* **264B** 62; 1991 *Mod. Phys. Lett.* A **6** 893

[11] Brzeziński T and Majid S 1993 *Commun. Math. Phys.* **157** 591

[12] Sun X D and Wang S K 1992 Bicovariant differential calculus on quantum group $GL_q(n)$, *Preprint* Worldlab-Beijing CCAST-92-04

[13] Aschieri P and Castellani L 1993 *Int. J. Mod. Phys.* A **8** 1667

[14] Castellani L 1992 *Phys. Lett.* **292B** 93; 1992 $U_q(N)$ gauge theories, *Preprint* DFTT-74/92

[15] Watamura S 1993 *Commun. Math. Phys.* **158** 67

[16] Brzeziński T and Majid S 1993 *Phys. Lett.* **298B** 339

[17] Becchi C, Rouet A and Stora R 1976 *Ann. Phys.* **98** 287; Tyuitin I V 1975 Lebedev *Preprint* FIAN **39** unpublished; Kugo T and Ojima 1979 I *Prog. Theor. Phys. Suppl.* **66** 1

[18] Chern S S 1979 *Complex Manifolds Without Potential Theory* (especially the appendix: Geometry of characteristic classes) (New York: Springer)

[19] B 1983 Chiral anomalies and differential geometry, Lectures given at Les Houches, *Preprint* LBL-16747; Mañes J, Stora R and Zumino B 1985 *Commun. Math. Phys.* **102** 157

[20] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312; Baxter R J 1972 *Ann. Phys.* **70** 193

[21] Zhong-Qi Ma 1993 *Yang–Baxter Equation and Quantum Enveloping Algebras* (Singapore: World Scientific)

[22] Faddeev L D, Reshetikhin N Y and Takhtajan L A 1988 Quantization of Lie groups and Lie algebras, *Algebraic Analysis* (New York: Academic) p 129

[23] D Gross and R Jackiw 1972 *Phys. Rev.* D **6** 477; Jackiw R 1985 New directions for topological research on gauge theories, *Symposium on Anomalies, Geometry, Topology* ed W A Bardeen and A R White (Singapore: World Scientific) p 159

[24] Gasper G and Rahman M 1990 Basic hypergeometric series, *Encyclopedia of Mathematics and its Applications* vol 35 Cambridge; Jackson F H 1910 *Quart. J. Pure Appl. Math.* **41** 193

[25] Bo-Yu Hou and Yao-Zhong Zhang 1987 *J. Math. Phys.* **28** 1709